

# FRACTIONAL DYNAMICAL SYSTEMS ON FRACTIONAL LEIBNIZ ALGEBROIDS

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*Dedicated to Acad. Prof. Dr. Radu Miron at his 80 th anniversary*

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**Abstract.** In this paper we consider the fractional tangent bundle on a differentiable manifold. A fractional Leibniz structure on an algebroid is defined. The fractional dynamical system on a fractional Leibniz algebroid is defined and it is discussed. Some illustrative examples are presented.<sup>1</sup>

## 1 Introduction

The theory of derivative of noninteger order goes back to Leibniz, Liouville, Riemann, Grunwald and Letnikov. Derivatives of fractional order have found many applications in recent studies in mechanics, physics, economics, medicine. Classes of fractional differentiable systems have studied in [10], [4].

In the first section the fractional tangent bundle to a differentiable manifold is defined, using the method of Radu Miron's from [8]. In this paper the fractional dynamical systems on fractional Leibniz algebroids are presented. The associated geometrical objects have an geometric character. Also, some examples for fractional dynamical systems of this type are given.

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## 2 Fractional tangent bundle on a manifold

Let  $f : [a, b] \rightarrow \mathbf{R}$  and  $\alpha \in \mathbf{R}, \alpha > 0$ . The *Riemann - Liouville fractional derivative at to left of a*, respectively *at to right of b* is the function  $f \rightarrow_a D_t^\alpha f$  resp.  $f \rightarrow_t D_b^\alpha f$ , where:

$$\begin{cases} {}_a D_t^\alpha f(t) &= \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t (t-s)^{m-\alpha-1} (f(s) - f(a)) ds \\ {}_t D_b^\alpha f(t) &= \frac{1}{\Gamma(m-\alpha)} \left(-\frac{d}{dt}\right)^m \int_t^b (t-s)^{m-\alpha-1} (f(s) - f(b)) ds, \end{cases} \quad (1)$$

where  $m \in \mathbf{N}^*$  such that  $m-1 \leq \alpha \leq m$ ,  $\Gamma$  is the Euler gamma function and  $\left(\frac{d}{dt}\right)^m = \frac{d}{dt} \circ \frac{d}{dt} \circ \dots \circ \frac{d}{dt}$ .

We will denote sometimes  $D_t^\alpha = {}_a D_t^\alpha$ .

The following proposition holds.

**Proposition 2.1** ([3]) (i) If  $\lim_{n \rightarrow \infty} \alpha_n = p \in \mathbf{N}^*$ , then:

$$\lim_{n \rightarrow \infty} ({}_a D_t^{\alpha_n} f(t)) = D_t^p f(t), \quad \lim_{n \rightarrow \infty} ({}_t D_b^{\alpha_n} f(t)) = D_t^p f(t).. \quad (2)$$

(ii) If  $f(t) = c, (\forall) t \in [a, b]$ ,  $D_t^\alpha f(t) = 0$ .

(iii) If  $f_1(t) = t^\gamma, (\forall) t \in [a, b]$ , then  $D_t^\alpha f_1(t) = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}$ .

(iv) If  $f_1, f_2$  are analytical functions on  $(a, b)$ , then:

$$D_t^\alpha (f_1 f_2)(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} D_t^{\alpha-k} f_1(t) \left(\frac{d}{dt}\right)^k f_2(t). \quad (3)$$

(v) If  $f : [a, b] \rightarrow \mathbf{R}$  is analytical function on  $(a, b)$ , and  $0 \in (a, b)$  then:

$$f(t) = \sum_{h=0}^{\infty} E_{\alpha, h}(t) D_t^{\alpha h} f(t)|_{t=0}, \quad (4)$$

where  $E_{\alpha, h}$  is the Mittag - Leffler 's function:

$$E_{\alpha, h}(t) = \sum_{h=0}^{\infty} \frac{1}{\Gamma(1 + \alpha h)} t^{\alpha h}. \quad (5)$$

□

Let  $\alpha \in \mathbf{R}, \alpha > 0$  and  $M$  be a manifold of dimension  $n$  and  $U$  a local chart on  $M$ . We say that the curves  $c_1, c_2 : I \rightarrow M, 0 \in I, c_1(0) = c_2(0) \in M$  have *fractional contact  $\alpha$  in  $x_0$* , if for all  $f \in C^\infty(U), x_0 \in U$ , the following relation holds:

$$D_t^\alpha (f \circ c_1)|_{t=0} = D_t^\alpha (f \circ c_2)|_{t=0}. \quad (6)$$

The set of equivalence classes  $([c]_{x_0}^\alpha)$  is called the *fractional tangent space* in  $x_0$  and it is denoted by  $T_{x_0}^\alpha(U)$ .

Let  $T^\alpha(M) = \bigcup_{x_0 \in M} T_{x_0}^\alpha(U)$  and the projection  $\pi^\alpha : T^\alpha(M) \rightarrow M$  given by  $\pi^\alpha([c]_{x_0}^\alpha) = x_0$ .

On  $T^\alpha(M)$  there exists a differentiable structure and we can prove that  $(T^\alpha(M), \pi^\alpha, M)$  is a differentiable bundle.

In a system of local coordinates on  $M$ , if  $x_0 \in U$  and  $c : I \rightarrow M$  is a curve given by  $x^i = x^i(t)$ ,  $(\forall) t \in I$ , the class  $([c]_{x_0}^\alpha)$  is given by:

$$x^i(t) = x^i(0) + \frac{1}{\Gamma(1+\alpha)} t^\alpha D_t^\alpha x^i(t)|_{t=0}, \quad t \in (\varepsilon, \varepsilon). \quad (7)$$

On the open set  $(\pi^\alpha)^{-1}(U) \in T^\alpha(M)$ , the local coordinates of the element  $([c]_{x_0}^\alpha)$  are  $(x^i, y^{i(\alpha)})$ , where:

$$x^i = x^i(0), \quad y^{i(\alpha)} = \frac{1}{\Gamma(1+\alpha)} D_t^\alpha x^i(t), i = \overline{1, n}. \quad (8)$$

**Proposition 2.2** ([1], [2]) *Let  $U, \overline{U}$  be two local charts on  $M$  such that  $U \cap \overline{U} \neq \emptyset$  and*

$$\overline{x}^i = \overline{x}^i(x^1, x^2, \dots, x^n), \quad \det\left(\frac{\partial \overline{x}^i}{\partial x^j}\right) \neq 0, \quad i = \overline{1, n} \quad (9)$$

*the coordinate transformations. The coordinate transformations on  $(\pi^\alpha)^{-1}(U \cap \overline{U})$  are given by:*

$$\overline{x}^i = \overline{x}^i(x^1, x^2, \dots, x^n), \quad \overline{y}^{i(\alpha)} = J_j^{\alpha i}(x, \overline{x}) y^{j(\alpha)}, \quad (10)$$

*where:*

$$J_j^{\alpha i}(x, \overline{x}) = \frac{1}{\Gamma(1+\alpha)} D_{\overline{x}^j}^\alpha (x^i)^\alpha \quad (11)$$

*and  $D_{x^i}^\alpha$  is defined by:*

$$D_{x^i}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x^i} \cdot \int_{a^i}^{x^i} \frac{f(x^1, \dots, x^{i-1}, s, x^{i+1}, \dots, x^n) - f(x^1, \dots, x^{i-1}, a^i, x^{i+1}, \dots, x^n)}{(x^i - s)^\alpha} ds, \quad (12)$$

*with  $U_{ab} = \{x \in U, a^i \leq x^i \leq b^i, i = \overline{1, n}\} \subseteq U$ .  $\square$*

Let  $\mathcal{D}^1(U)$  the module of 1 - forms defined on  $U \subseteq M$ . Using the fractional exterior derivative  $d^\alpha : C^\infty(U) \rightarrow \mathcal{D}^1(U)$ ,  $f \rightarrow d^\alpha(f)$  ( see [2] ), where  $d^\alpha(f)$  is given by:

$$d^\alpha(f) = d(x^i)^\alpha D_{x^i}^\alpha(f) \quad (13)$$

follows:

$$\begin{cases} d(x^i)^\alpha &= J_j^{\alpha i}(x, \bar{x}) d(\bar{x}^j)^\alpha \\ D_{x^i}^\alpha &= J_i^{\alpha j}(\bar{x}, x) D_{\bar{x}^j}^\alpha \\ J_k^{\alpha i}(x, \bar{x}) J_j^{\alpha k}(\bar{x}, x) &= \delta_j^i. \end{cases} \quad (14)$$

We denote by  $\mathcal{X}^\alpha(U)$  the module of fractional vector fields generated by the operators  $\{D_{x^i}^\alpha, i = \overline{1, n}\}$ . A fractional vector field  $\overset{\alpha}{X} \in \mathcal{X}^\alpha(U)$  has the following form:

$$\overset{\alpha}{X} = \overset{\alpha}{X} D_{x^i}^\alpha, \quad \overset{\alpha}{X} \in C^\infty(U), i = \overline{1, n}, \quad (15)$$

which for a change of local charts, the correspondent components satisfies the relations:

$$\frac{\overset{\alpha}{X}}{\bar{X}} = J_j^{\alpha i}(x, \bar{x}) \frac{\overset{\alpha}{X}}{\bar{X}}, \quad i, j = \overline{1, n}. \quad (16)$$

The fractional differentiable equations associated to fractional vector field  $\overset{\alpha}{X}$  is:

$$D_t^\alpha x^i(t) = \overset{\alpha}{X}^i(x(t)), \quad i = \overline{1, n} \quad (17)$$

or equivalently ( using the notation (8)):

$$\Gamma(1 + \alpha) y^{i(\alpha)}(t) = \overset{\alpha}{X}^i(x(t)), \quad i = \overline{1, n}. \quad (18)$$

The fractional differential equations (17) with initial conditions have solutions, see [3]. Examples of fractional differentiable equations on  $\mathbf{R}$  can be find in [4].

### 3 Fractional Leibniz dynamical systems

Let the module  $\mathcal{X}^\alpha(U)$  of fractional vector fields generated by the operators  $\{D_{x^i}^\alpha, i = \overline{1, n}\}$  and the module  $\mathcal{D}^\alpha(U)$  generated by the 1- forms  $\{d(x^i)^\alpha, i = \overline{1, n}\}$ . Applying the Proposition 2.2 it follows:

$$(d(x^i)^\alpha)(D_{x^j}^\alpha) = D_{x^j}^\alpha(x^i)^\alpha = \Gamma(1 + \alpha) \delta_j^i. \quad (19)$$

If  $\overset{\alpha}{X} \in \mathcal{X}^\alpha(U)$  and  $\overset{\alpha}{\omega} \in \mathcal{D}^\alpha(U)$  such that  $\overset{\alpha}{\omega} = \overset{\alpha}{\omega}_i d(x^i)^\alpha$ , then  $\overset{\alpha}{\omega}(\overset{\alpha}{X}) = \Gamma(1 + \alpha) \overset{\alpha}{X}^i \overset{\alpha}{\omega}_i$ .

Let be a fractional 2- contravariant tensor field  $\overset{\alpha}{B} \in \mathcal{X}^\alpha(U) \times \mathcal{X}^\alpha(U)$  and  $d^\alpha f, d^\alpha g \in \mathcal{D}^\alpha(U)$  defined by (13).

The bilinear map  $[\cdot, \cdot]^\alpha : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$  defined by:

$$[f, g]^\alpha = B(d^\alpha f, d^\alpha g), \quad (\forall) f, g \in C^\infty(M), \quad (20)$$

is called the *fractional Leibniz bracket*.

If  $\overset{\alpha}{B} = \overset{\alpha}{B}^{\alpha ij} D_{x^i}^\alpha \otimes D_{x^j}^\alpha$ , from (20) follows:

$$[f, g]^\alpha = \overset{\alpha}{B}^{\alpha ij} \cdot D_{x^i}^\alpha f \cdot D_{x^j}^\alpha g. \quad (21)$$

Since

$$D_{x^i}^\alpha(fh)(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (D_{x^i}^{\alpha-k} f(x)) \left(\frac{\partial}{\partial x^i}\right)^k h(x), \quad (22)$$

it follows

$$[fh, g]^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} \cdot \overset{\alpha}{B}^{\alpha ij} (D_{x^i}^{\alpha-k} f) \cdot \left(\frac{\partial}{\partial x^i}\right)^k h \cdot D_{x^j}^\alpha g, \quad (23)$$

Similarly, one obtain

$$[f, gh]^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} \cdot \overset{\alpha}{B}^{\alpha ij} (D_{x^i}^\alpha f) \cdot D_{x^j}^{\alpha-k} (g) \cdot \left(\frac{\partial}{\partial x^i}\right)^k h. \quad (24)$$

The pair  $(M, [\cdot, \cdot]^\alpha)$  is called *fractional Leibniz manifold*. If the bracket  $[\cdot, \cdot]^\alpha$  is skew-symmetric, that is  $[f, g]^\alpha = -[g, f]^\alpha$  for all  $f, g \in C^\infty(M)$  we say that  $(M, [\cdot, \cdot]^\alpha)$  is a *fractional almost Poisson manifold*. If  $\alpha \rightarrow 1$ , then one obtain the concept from [6].

For  $h \in C^\infty(M)$ , the fractional vector field  $\overset{\alpha}{X}_h$  defined by

$$\overset{\alpha}{X}_h(f) = [f, h]^\alpha, \quad (\forall) f \in C^\infty(M), \quad (25)$$

is called the *fractional Leibniz vector field associated to h*. The fractional dynamical system associated to  $\overset{\alpha}{X}_h$  is called the *fractional Leibniz dynamical system*.

If  $(x^i), i = \overline{1, n}$  is a system of local coordinates on  $M$ , then the fractional Leibniz dynamical system is given by

$$D_t^\alpha x^i(t) = [x^i(t), h(t)]^\alpha, \quad \text{where} \quad [x^i, h]^\alpha = \overset{\alpha}{B}^{\alpha ij} \cdot D_{x^j}^\alpha h. \quad (26)$$

**Example 3.1.** Let the constant fractional 2- contravariant tensor  $\overset{\alpha}{g} = (\overset{\alpha}{g}{}^{ij})$  defined on  $\mathbf{R}^3$  by

$$\overset{\alpha}{g} = \begin{pmatrix} s_1\gamma_1 & 0 & 0 \\ 0 & s_2\gamma_2 & 0 \\ 0 & 0 & s_3\gamma_3 \end{pmatrix}, \quad (27)$$

where  $s_1, s_2, s_3 \in \{-1, 1\}$  and  $\gamma_1, \gamma_2, \gamma_3$  satisfies the relation  $\gamma_1 + \gamma_2 + \gamma_3 = 0$ . For  $h = x^1x^2x^3$ , the associated fractional Leibniz dynamical system is

$$\begin{cases} D_t^\alpha x^1 &= s_1\gamma_1 D_{x^1}^\alpha(h) &= \frac{\Gamma(2)}{\Gamma(2-\alpha)} s_1\gamma_1 x^2x^3(x^1)^{1-\alpha} \\ D_t^\alpha x^2 &= s_2\gamma_2 D_{x^2}^\alpha(h) &= \frac{\Gamma(2)}{\Gamma(2-\alpha)} s_2\gamma_2 x^1x^3(x^2)^{1-\alpha} \\ D_t^\alpha x^3 &= s_3\gamma_3 D_{x^3}^\alpha(h) &= \frac{\Gamma(2)}{\Gamma(2-\alpha)} s_3\gamma_3 x^1x^2(x^3)^{1-\alpha}. \end{cases} \quad (28)$$

If  $\alpha \rightarrow 1$ , it follows the system (7) in [10].

For  $\overset{\alpha}{h} = (x^1)^\alpha(x^2)^\alpha(x^3)^\alpha$ , the associated fractional Leibniz dynamical system is

$$\begin{cases} D_t^\alpha x^1 &= \Gamma(1+\alpha)s_1\gamma_1 x^2x^3 \\ D_t^\alpha x^2 &= \Gamma(1+\alpha)s_2\gamma_2 x^1x^3 \\ D_t^\alpha x^3 &= \Gamma(1+\alpha)s_3\gamma_3 x^1x^2. \end{cases} \quad (29)$$

□

Let  $\overset{\alpha}{P}$  be a skew-symmetric fractional 2- contravariant tensor field and a non-degenerate symmetric fractional 2- contravariant tensor field  $\overset{\alpha}{g}$  on the manifold  $M$ . We define the bracket  $[\cdot, \cdot]^\alpha : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$  by

$$[f, h]^\alpha = \overset{\alpha}{P}(d^\alpha f, d^\alpha h) + \overset{\alpha}{g}(d^\alpha f, d^\alpha h), \quad (\forall) f, h \in C^\infty(M). \quad (30)$$

The 4- tuple  $(M, \overset{\alpha}{P}, \overset{\alpha}{g}, [\cdot, \cdot]^\alpha)$  is called *fractional almost metric manifold*. The fractional dynamical system associated to  $h \in C^\infty(M)$  is

$$D_t^\alpha x^i(t) = [x^i(t), h(t)]^\alpha, \quad \text{where} \quad [x^i, h]^\alpha = \overset{\alpha}{P}{}^{ij} D_{x^j}^\alpha h + \overset{\alpha}{g}{}^{ij} D_{x^j}^\alpha h. \quad (31)$$

**Example 3.2.** Let be the fractional 2 - contravariant tensors fields  $\overset{\alpha}{P} = (\overset{\alpha}{P}{}^{ij}), \overset{\alpha}{g} = (\overset{\alpha}{g}{}^{ij})$  on  $\mathbf{R}^3$  and the function  $h \in C^\infty(\mathbf{R}^3)$  given by :

$$\overset{\alpha}{P} = \begin{pmatrix} 0 & x^3 & -x^2 \\ -x^3 & 0 & x^1 \\ x^2 & -x^1 & 0 \end{pmatrix},$$

$$\overset{\alpha}{g} = \begin{pmatrix} -a_2(x^2)^2 - a_3(x^3)^2 & a_1a_2x^1x^2 & a_1a_3x^1x^3 \\ a_1a_2x^1x^2 & -a_1(x^1)^2 - a_3(x^3)^2 & a_2a_3x^2x^3 \\ a_1a_3x^1x^3 & a_2a_3x^2x^3 & -a_1(x^1)^2 - a_2(x^2)^2 \end{pmatrix},$$

$$h = (a_1 + 1)(x^1)^\alpha + (a_2 + 1)(x^2)^\alpha + (a_3 + 1)(x^3)^\alpha.$$

Since  $D_{x^1}^\alpha h = (a_1 + 1)\Gamma(1 + \alpha)$ ,  $D_{x^2}^\alpha h = (a_2 + 1)\Gamma(1 + \alpha)$ ,  $D_{x^3}^\alpha h = (a_3 + 1)\Gamma(1 + \alpha)$ , the fractional Leibniz dynamical system (31) associated to  $h$  is

$$\begin{aligned} \begin{pmatrix} D_t^\alpha x^1 \\ D_t^\alpha x^2 \\ D_t^\alpha x^3 \end{pmatrix} &= \Gamma(1 + \alpha) \overset{\alpha}{P} \begin{pmatrix} a_1 + 1 \\ a_2 + 1 \\ a_3 + 1 \end{pmatrix} + \Gamma(1 + \alpha) \overset{\alpha}{g} \begin{pmatrix} a_1 + 1 \\ a_2 + 1 \\ a_3 + 1 \end{pmatrix} = \\ &= \Gamma(1 + \alpha) (\overset{\alpha}{P} + \overset{\alpha}{g}) \begin{pmatrix} a_1 + 1 \\ a_2 + 1 \\ a_3 + 1 \end{pmatrix}. \square \end{aligned}$$

Let be two fractional 2 - contravariant tensors fields  $\overset{\alpha}{P}$  and  $\overset{\alpha}{g}$  on  $M$ . Define the bracket  $[\cdot, (\cdot, \cdot)] : C^\infty(M) \times C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$  by:

$$[f, h]^\alpha = \overset{\alpha}{P}(d^\alpha f, d^\alpha h_1) + \overset{\alpha}{g}(d^\alpha f, d^\alpha h_2), \quad (\forall) f, h_1, h_2 \in C^\infty(M). \quad (32)$$

The fractional vector field  $\overset{\alpha}{X}_{h_1 h_2}$  defined by

$$\overset{\alpha}{X}_{h_1 h_2} = [f, (h_1, h_2)], \quad (\forall) f \in C^\infty(M). \quad (33)$$

is called the *fractional almost Leibniz vector field associate to the functions*  $h_1, h_2 \in C^\infty(M)$ . The dynamical system associated to  $\overset{\alpha}{X}_{h_1 h_2}$  is called the *fractional almost Leibniz dynamical system*.

Locally, the fractional almost Leibniz dynamical system is given by:

$$D_t^\alpha x^i(t) = \overset{\alpha}{P}{}^{\alpha ij} D_{x^j}^\alpha h_1 + \overset{\alpha}{g}{}^{\alpha ij} D_{x^j}^\alpha h_2. \quad (34)$$

**Example 2.3.** Let be the fractional 2 - contravariant tensors fields  $\overset{\alpha}{P} = (\overset{\alpha}{P}{}^{\alpha ij})$ ,  $\overset{\alpha}{g} = (\overset{\alpha}{g}{}^{\alpha ij})$  on  $\mathbf{R}^3$  and the functions  $h_1, h_2 \in C^\infty(\mathbf{R}^3)$  given by :

$$\overset{\alpha}{P} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & x^1 \\ 0 & -x^1 & 0 \end{pmatrix},$$

$$\overset{\alpha}{g} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(x^3)^2 & 0 \\ 0 & 0 & -(x^2)^2 \end{pmatrix},$$

$$\overset{\alpha}{h}_1 = (x^2)^{1+\alpha} + (x^3)^{1+\alpha}, \quad \overset{\alpha}{h}_2 = (x^1)^{1+\alpha} + (x^3)^\alpha.$$

Since

$$D_{x^1}^\alpha \overset{\alpha}{h}_1 = 0, \quad D_{x^2}^\alpha \overset{\alpha}{h}_1 = \Gamma(1+\alpha)x^2, \quad D_{x^3}^\alpha \overset{\alpha}{h}_1 = \Gamma(1+\alpha)x^3;$$

$$D_{x^1}^\alpha \overset{\alpha}{h}_2 = \Gamma(1+\alpha)x^1, \quad D_{x^2}^\alpha \overset{\alpha}{h}_2 = 0, \quad D_{x^3}^\alpha \overset{\alpha}{h}_2 = \Gamma(1+\alpha),$$

the system (34) becomes:

$$\begin{pmatrix} D_t^\alpha x^1 \\ D_t^\alpha x^2 \\ D_t^\alpha x^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & x^1 \\ 0 & -x^1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \Gamma(1+\alpha)x^2 \\ \Gamma(1+\alpha)x^3 \end{pmatrix} + \\ + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(x^3)^2 & 0 \\ 0 & 0 & -(x^2)^2 \end{pmatrix} \begin{pmatrix} \Gamma(1+\alpha)x^1 \\ 0 \\ \Gamma(1+\alpha) \end{pmatrix}$$

or equivalently

$$\begin{cases} D_t^\alpha x^1 &= \Gamma(1+\alpha)x^2 \\ D_t^\alpha x^2 &= \Gamma(1+\alpha)x^1x^3 \\ D_t^\alpha x^3 &= -\Gamma(1+\alpha)x^1x^2 - \Gamma(1+\alpha)(x^2)^2. \end{cases} \quad (35)$$

The system (35) is called the *fractional Maxwell- Bloch equations*.

If in (35) we take  $\alpha \rightarrow 1$ , then one obtain the Maxwell-Bloch equations.

## 4 Fractional Leibniz algebroids

Let  $M$  be a smooth manifold of dimension  $n$ , let  $\pi : E \rightarrow M$  be a vector bundle and  $\pi^* : E^* \rightarrow M$  the dual vector bundle. By  $Sec(M, E)$  or  $Sec(\pi)$  we denote the sections of  $\pi$ .

A *fractional Leibniz algebroid structure* on a vector bundle  $\pi : E \rightarrow M$  is given by a bracket ( bilinear operation )  $[\cdot, \cdot]^\alpha$  on the space of sections  $Sec(\pi)$  and two vector bundle morphisms  $\overset{\alpha}{\rho}_1, \overset{\alpha}{\rho}_2 : E \rightarrow T^\alpha M$  ( called the *left*



and the *right fractional anchor* , respectively ) such that

$$\begin{cases} [e_a, e_b]^\alpha = C_{ab}^c e_c \\ [f\sigma_1, g\sigma_2]^\alpha = f\rho_1^\alpha(\sigma_1)(g)\sigma_2 - g\rho_2^\alpha(\sigma_2)(f)\sigma_1 + fg[\sigma_1, \sigma_2]^\alpha \end{cases} \quad (36)$$

for all  $\sigma_1, \sigma_2 \in \text{Sec}(\pi)$  and  $f, g \in C^\infty(M)$ .

A vector bundle  $\pi : E \rightarrow M$  endowed with a fractional Leibniz algebroid structure  $([\cdot, \cdot]^\alpha, \rho_1^\alpha, \rho_2^\alpha)$  on  $E$  , is called *fractional Leibniz algebroid* over  $M$  and denoted by  $(E, [\cdot, \cdot]^\alpha, \rho_1^\alpha, \rho_2^\alpha)$ .

A fractional Leibniz algebroid with an antisymmetric bracket  $[\cdot, \cdot]^\alpha$  ( in this case we have  $\rho_1^\alpha = -\rho_2^\alpha$  ) is called *fractional pre - Lie algebroid*.

In a system of local coordinates the relation (36) reads:

$$[\sigma_1^a e_a, \sigma_2^b e_b]^\alpha = \sigma_1^a \rho_1^\alpha(e_a)(\sigma_2^b) e_b - \sigma_2^b \rho_2^\alpha(e_a)(\sigma_1^a) e_b + \sigma_1^a \sigma_2^b C_{ab}^c e_c. \quad (37)$$

If  $\rho_1^\alpha(e_a) = \rho_{1a}^{\alpha i} D_{x^i}^\alpha$ ,  $\rho_2^\alpha(e_b) = \rho_{2b}^{\alpha i} D_{x^i}^\alpha$ , from (37) follows:

$$[\sigma_1^a e_a, \sigma_2^b e_b]^\alpha = \sigma_1^a \rho_{1a}^{\alpha i} (D_{x^i}^\alpha \sigma_2^b) e_b - \sigma_2^b \rho_{2a}^{\alpha i} (D_{x^i}^\alpha \sigma_1^a) e_b + \sigma_1^a \sigma_2^b C_{ab}^c e_c. \quad (38)$$

In the following, we establish a correspondence between the fractional Leibniz algebroid structures on the vector bundle  $\pi : E \rightarrow M$  and the fractional 2- contravariant tensor fields on bundle manifold  $E^*$  of the dual vector bundle  $\pi^* : E^* \rightarrow M$ .

For a given section  $\sigma \in \text{Sec}(\pi)$ , we define the function  $i_{E^*}\sigma$  on  $E^*$  by the relation :

$$i_{E^*}\sigma(a) = \langle \sigma(\pi^*(a)), a \rangle, \quad \text{for } a \in E^*, \quad (39)$$

where  $\langle \cdot, \cdot \rangle$  is the canonical pairing between  $E$  and  $E^*$ . If  $\sigma = \sigma^a e_a$  and  $a \in E^*$  has the coordinates  $(x^i, \xi_a)$ , then:

$$i_{E^*}\sigma(a) = \sigma^a \xi_a. \quad (40)$$

Let  $\hat{\Lambda}^\alpha$  be a fractional 2 - contravariant tensor field on  $E^*$  and the bracket  $[\cdot, \cdot]_{\hat{\Lambda}^\alpha}^\alpha$  of functions defined by:

$$[f, g]_{\hat{\Lambda}^\alpha}^\alpha = \hat{\Lambda}^{\alpha\beta} (d^{\alpha\beta} f, d^{\alpha\beta} g), \quad (\forall) f, g \in C^\infty(E^*), \quad (41)$$

where

$$d^{\alpha\beta} f = d(x^i)^\alpha D_{x^i}^\alpha f + d(\xi_a)^\beta D_{\xi_a}^\beta f = d^\alpha(f) + d^\beta(f). \quad (42)$$

In the basis  $\{D_{x^i}^\alpha, D_{\xi_a}^\beta\}, i = \overline{1, n}, a = \overline{1, m}$  of the module  $\mathcal{X}^{\alpha\beta}(\pi^{*-1}(U))$ , the components  $\Lambda^{\alpha\beta}$  are given by:

$$\Lambda^{\alpha\beta} = A_{ab}D_{\xi_a}^\beta \otimes D_{\xi_b}^\beta + A_{1a}^i D_{\xi_a}^\beta \otimes D_{x^i}^\alpha + A_{2a}^i D_{x^i}^\alpha \otimes D_{\xi_a}^\beta. \quad (43)$$

For a given fractional 2 - contravariant tensor field  $\overset{\alpha\beta}{\Lambda}$  on  $E^*$ , we say that  $\overset{\alpha\beta}{\Lambda}$  is *linear*, if for each pair  $(\mu_1, \mu_2)$  of sections of  $\pi^*$ , the function  $\overset{\alpha\beta}{\Lambda}(d(i_{E^*}\mu_1)^\beta, d(i_{E^*}\mu_2)^\beta)$  defined on  $E^*$  is linear with respect the coordinates  $\xi_a$ .

If  $\mu_1 = \mu_1^a(x)e_a, \mu_2 = \mu_2^a(x)e_a$ , then  $d_{E^*}\mu_1 = \mu_1^a(x)\xi_a, d_{E^*}\mu_2 = \mu_2^a(x)\xi_a$  and  $\overset{\alpha\beta}{\Lambda}(d(i_{E^*}\mu_1)^\beta, d(i_{E^*}\mu_2)^\beta) = A_{ab}(x, \xi)(\mu_1^c(x))^\alpha(\mu_2^a(x))^\alpha D_{\xi_a}^\beta(\xi_c)D_{\xi_b}^\beta(\xi_a)^\beta = \frac{1}{\Gamma(1+\alpha)^2}(\mu_1^a(x))^\beta(\mu_2^b(x))^\beta A_{ab}(x, \xi)$ .

It follows that  $\overset{\alpha\beta}{\Lambda}$  is linear if and only if  $A_{ab}(x, \xi) = C_{ab}^c(x)\xi_c$ .

The fractional formulation of the Grabowski and Urbanski's Theorem from [6], is the following.

**Theorem 4.1.** *For every fractional Leibniz algebroid structure on  $\pi : E \rightarrow M$  with the bracket  $[\cdot, \cdot]^\alpha$  and the fractional anchors  $\overset{\alpha}{\rho}_1, \overset{\alpha}{\rho}_2$ , there exists an unique fractional 2 - contravariant tensor field  $\overset{\alpha}{\Lambda}$  on  $E^*$  such that the following relations hold:*

$$\begin{cases} i_{E^*}[\sigma_1, \sigma_2] &= [(i_{E^*}\sigma_1)^\beta, (i_{E^*}\sigma_2)^\beta]_{\overset{\alpha}{\Lambda}}^{\alpha\beta} \\ \pi^*(\overset{\alpha}{\rho}_1(\sigma)(f)) &= [(i_{E^*}\sigma)^\beta, \pi^*f]_{\overset{\alpha}{\Lambda}}^{\alpha\beta} \\ \pi^*(\overset{\alpha}{\rho}_2(\sigma)(f)) &= [\pi^*f, (i_{E^*}\sigma)^\beta]_{\overset{\alpha}{\Lambda}}^{\alpha\beta}, \end{cases} \quad (44)$$

for all  $\sigma, \sigma_1, \sigma_2 \in \text{Sec}(\pi)$  and  $f \in C^\infty(M)$ .

Conversely, every fractional 2 - contravariant linear tensor field  $\overset{\alpha\beta}{\Lambda}$  on  $E^*$  defines a fractional Leibniz algebroid on  $E$  if the relations (44) hold.  $\square$

Let  $(x^i), i = \overline{1, n}$  be a local coordinate system on  $U \subseteq M$  and let  $\{e_1, \dots, e_m\}$  be a basis of local sections of  $E|_U$  ( $\dim M = n, \dim E = n + m$ ). We denote by  $\{e^1, \dots, e^m\}$  the dual basis of local sections of  $E^*|_U$  and  $(x^i, y^a)$  ( resp.,  $(x^i, \xi_a)$  ) the corresponding coordinate system on  $E$  ( resp.,  $E^*$  ).

Let  $\overset{\alpha\beta}{\Lambda}$  given by (43). Using (44), it is easy to see that every linear fractional 2 - contravariant tensor field  $\overset{\alpha\beta}{\Lambda}$  on  $E^*$  has the form:

$$\overset{\alpha\beta}{\Lambda} = C_{ab}^d \xi_d D_{\xi_a}^\beta \otimes D_{\xi_b}^\beta + \overset{\alpha i}{\rho}_{1a} D_{\xi_a}^\beta \otimes D_{x^i}^\alpha - \overset{\alpha i}{\rho}_{2a} D_{x^i}^\alpha \otimes D_{\xi_a}^\beta, \quad (45)$$

where  $C_{ab}^d, \rho_{1a}^{\alpha i}, \rho_{2a}^{\alpha i} \in C^\infty(M)$  are functions of  $x^i$ .

The correspondence between  $\Lambda_{\alpha\beta}$  and a fractional Leibniz algebroid structure is given by the following relations :

$$[e_a, e_b]^\alpha = C_{ab}^d e_d, \quad \rho_1^\alpha(e_a) = \rho_{1a}^{\alpha i} D_{x^i}^\alpha, \quad \rho_2^\alpha(e_a) = \rho_{2a}^{\alpha i} D_{x^i}^\alpha. \quad (46)$$

We call a *fractional dynamical system on the fractional Leibniz algebroid*  $\pi : E \rightarrow M$ , the fractional dynamical system associated to vector field  $X_h^{\alpha\beta}$  with  $h \in C^\infty(E^*)$  given by:

$$X_h^{\alpha\beta}(f) = \Lambda^{\alpha\beta}(d^{\alpha\beta}f, d^{\alpha\beta}h), \quad \text{for all } f \in C^\infty(E^*). \quad (47)$$

In a system of local coordinates  $(x^i, \xi_a)$  on  $E^*$ , the dynamical system (47) is given by :

$$\begin{cases} D_t^\alpha \xi_a = [\xi_a, h]_{\Lambda^{\alpha\beta}} \\ D_t^\alpha x^i = [x^i, h]_{\Lambda^{\alpha\beta}} \end{cases}. \quad (48)$$

where

$$\begin{cases} [\xi_a, h]_{\Lambda^{\alpha\beta}} = C_{ab}^d \xi_d D_{\xi_b}^\beta h + \rho_{1a}^{\alpha i} D_{x^i}^\alpha h \\ [x^i, h]_{\Lambda^{\alpha\beta}} = -\rho_{2a}^{\alpha i} D_{\xi_a}^\beta h \end{cases}. \quad (49)$$

If  $\alpha \rightarrow 1, \beta \rightarrow 1$ , dynamical system (48) was studied in [6].

If  $\alpha \rightarrow 1$  dynamical system (48) has the form:

$$\begin{cases} \dot{x}^i = -\rho_{2a}^{\alpha i} D_{\xi_a}^\beta h \\ D_t^\beta \xi_a = C_{ab}^d \xi_d D_{\xi_b}^\beta h + \rho_{1a}^{\alpha i} \frac{\partial h}{\partial x^i} \end{cases}. \quad (50)$$

If  $\beta \rightarrow 1$  dynamical system (48) has the form:

$$\begin{cases} D_t^\alpha x^i = -\rho_{2a}^{\alpha i} \frac{\partial h}{\partial \xi_a} \\ \dot{\xi}_a = C_{ab}^d \xi_d \frac{\partial h}{\partial \xi_b} + \rho_{1a}^{\alpha i} D_{x^i}^\alpha h \end{cases}. \quad (51)$$

If the fractional Leibniz algebroid is a fractional pre - Lie algebroid ( that is,  $C_{ab}^d = -C_{ba}^d$  ), then the fractional dynamical system (48) is given by :

$$\begin{cases} D_t^\beta \xi_a = C_{ab}^d \xi_d D_{\xi_b}^\beta h + \rho_{1a}^{\alpha i} D_{x^i}^\alpha h \\ D_t^\alpha x^i = -\rho_{1a}^{\alpha i} D_{\xi_a}^\beta h \end{cases}. \quad (52)$$

If the fractional Leibniz algebroid is a fractional symmetric algebroid ( that is,  $C_{ab}^d = C_{ba}^d$  ), then the fractional dynamical system (48) is given by:

$$\begin{cases} D_t^\beta \xi_a &= C_{ab}^d \xi_d D_{\xi_a}^\beta h + \rho_{1a}^{\alpha i} D_{x^i}^\alpha h \\ D_t^\alpha x^i &= \rho_{1a}^{\alpha i} D_{\xi_a}^\beta h \end{cases}. \quad (53)$$

**Example 4.1.** Let the vector bundle  $\pi : E = \mathbf{R}^3 \times \mathbf{R}^3 \rightarrow \mathbf{R}^3$  and  $\pi^* : E^* = \mathbf{R}^3 \times (\mathbf{R}^3)^* \rightarrow \mathbf{R}^3$  the dual vector bundle. We consider on  $E^*$  the fractional 2 - contravariant linear tensor field  $\overset{\alpha}{\Lambda}$  defined by the matrix  $P^\beta$ , the fractional anchors  $\overset{\alpha}{\rho}_1, \overset{\alpha}{\rho}_2$  and the function  $h$  given by:

$$P^\beta = \begin{pmatrix} 0 & -\xi_3 x^3 & \xi_2 x^2 \\ \xi_3 x^3 & 0 & -\xi_1 x^1 \\ -\xi_2 x^2 & \xi_1 x^1 & 0 \end{pmatrix}, \quad \overset{\alpha}{\rho}_1 = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & 0 \\ -x^2 & 0 & 0 \end{pmatrix},$$

$$\overset{\alpha}{\rho}_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -x^1 \\ 0 & x^1 & 0 \end{pmatrix} \text{ and } h(x, \xi) = (x^2)^\alpha (\xi_2)^\beta + (x^3)^\alpha (\xi_3)^\beta, \alpha > 0, \beta > 0.$$

Using the calculus formulas:

$$D_{\xi_a}^\beta (\xi_b)^\gamma = \delta_b^a \xi_a^{\gamma-\beta} \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\beta)}, \quad D_{x^i}^\alpha (x^j)^\gamma = \delta_i^j (x^i)^{\gamma-\alpha} \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}$$

follows:

$$\begin{cases} D_{\xi_1}^\beta h = 0, & D_{\xi_2}^\beta h = \Gamma(1+\beta)(x^2)^\alpha, & D_{\xi_3}^\beta h = \Gamma(1+\beta)(x^3)^\alpha \\ D_{x^1}^\alpha h = 0, & D_{x^2}^\alpha h = \Gamma(1+\alpha)(\xi_2)^\beta, & D_{x^3}^\alpha h = \Gamma(1+\alpha)(\xi_3)^\beta. \end{cases}$$

The fractional dynamical system ( 48 ) for the given elements, has the following matrix form:

$$\begin{pmatrix} D_t^\beta \xi_1 \\ D_t^\beta \xi_2 \\ D_t^\beta \xi_3 \end{pmatrix} = \Gamma(1+\beta) \begin{pmatrix} 0 & -\xi_3 x^3 & \xi_2 x^2 \\ \xi_3 x^3 & 0 & -\xi_1 x^1 \\ -\xi_2 x^2 & \xi_1 x^1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ (x^2)^\alpha \\ (x^3)^\alpha \end{pmatrix} +$$

$$+ \Gamma(1+\alpha) \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & 0 \\ -x^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ (\xi_2)^\beta \\ (\xi_3)^\beta \end{pmatrix},$$

$$\begin{pmatrix} D_t^\alpha x^1 \\ D_t^\alpha x^2 \\ D_t^\alpha x^3 \end{pmatrix} = -\Gamma(1+\beta) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -x^1 \\ 0 & x^1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ (x^2)^\alpha \\ (x^3)^\alpha \end{pmatrix}.$$

From the above matrix equations follows the fractional dynamical system:

$$\left\{ \begin{array}{l} D_t^\beta \xi_1 = \Gamma(1+\beta)(-\xi_3(x^2)^\alpha x^3 + \xi_2 x^2 (x^3)^\alpha) + \\ \quad + \Gamma(1+\alpha)(-x^3(\xi_2)^\beta + x^2(\xi_3)^\beta) \\ D_t^\beta \xi_2 = -\Gamma(1+\beta)\xi_1(x^3)^\alpha \\ D_t^\beta \xi_3 = -\Gamma(1+\beta)\xi_1(x^2)^\alpha \\ D_t^\beta \xi_3 = \Gamma(1+\beta)\xi_1(x^2)^\alpha \\ D_t^\alpha x^1 = -\Gamma(1+\beta)(x^2)^\alpha \\ D_t^\alpha x^2 = -\Gamma(1+\beta)x^1(x^3)^\alpha \\ D_t^\alpha x^3 = \Gamma(1+\beta)x^1(x^3)^\alpha \end{array} \right. . \quad (54)$$

The fractional dynamical system (54) is the  $(\alpha, \beta)$ - *fractional dynamical system* associated to fractional Maxwell-Bloch equations.  $\square$

**Conclusion.** The numerical integration of the fractional systems presented in this paper will be discussed in future papers.

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